Expander Graphs

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Chapter 1

Preliminaries

This blueprint follows Fan Chung: Spectral Graph Theory (first published: AMS, 1992). Unless otherwise specified, all matrices are $n \times n$ where n is the number of vertices in G. We omit natural language proofs for theorems that only involve matrix multiplication and basic operations, although we formally prove them in Lean.

1.1 Weighted Graph

Definition 1 (Weighted Graph)**.** We extend the definition of a general graph to that of a weighted graph by defining a weight function over its edges.

$$\omega : \beta \to \mathbf{R}_+$$

with the condition that

$$e \in edgeSet \iff 0 < w(e)$$

We also add an arbitray orientation for undirected edges.

Definition 2. In a graph G, let d_v denote the degree (number of incident edges) of vertex v.

Definition 3. For $k \in \mathbb{N}$, a graph is said to be k-regular if every vertex v has degree $d_v = k$.

Definition 4. The volume of a graph is defined as the sum of the degrees of its vertices.

Definition 5. A vertex v is said to be isolated if $d_v = 0$.

Definition 6. A graph is said to be nontrivial if it contains at least one edge.

1.2 Matrices and Operators

Definition 7. The adjacency matrix of a graph G, denoted A is defined as follows,

$$A(u, v) = \begin{cases} 1 & \text{if } u \text{ and } v \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 8. Consider the matrix L, defined as follows

$$L(u,v) = \begin{cases} d_v & \text{if } u = v, \\ -1 & \text{if } u \text{ and } v \text{ are ajacent}, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 9 (Laplacian Matrix)**.** Consider the matrix $\mathcal{L}$, defined as follows

$$\mathcal{L}(u,v) = \begin{cases} 1 & \text{if } u = v \text{ and } d_v \neq 0, \\ -\dfrac{1}{\sqrt{d_u d_v}} & \text{if } u \text{ and } v \text{ are adjacent}, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 10. Let T denote the diagonal matrix with the (u,v)-th entry having value d_v.

Definition 11. We also define $T^{-1/2}$ with the convention that $T^{-1/2} = 0$ when $d_v = 0$.

Lemma 12 (Laplacian symmetric normalization)**.**

$$\mathcal{L} = T^{-1/2}\ L\ T^{-1/2}$$

Definition 13. The Laplacian of a graph can be viewed as an operator on the space of functions $g : V(G) \to \mathbb{R}$.

Definition 14. The Laplacian Operator satisfies

$$\mathcal{L}g(u) = \frac{1}{\sqrt{d_u}} \sum_{\substack{v \\ u \sim v}} \left(\frac{g(u)}{\sqrt{d_u}} - \frac{g(v)}{\sqrt{d_v}} \right).$$

Lemma 15. *When G is k-regular,*

$$\mathcal{L} = I - \frac{1}{k}A,$$

where $k > 0$, A is the adjacency matrix of G, and I is the identity matrix.

Lemma 16. *For a general graph without isolated vertices, we have*

$$\mathcal{L} = I - T^{-1/2} A T^{-1/2}.$$

Definition 17 (Boundary Operator)**.** Let S denote the matrix whose rows are indexed by the vertices and whose columns are indexed by the edges of G. Each column corresponding to an edge $e = \{u, v\}$ has an entry $1/\sqrt{d_u}$ in the row corresponding to u, an entry $-1/\sqrt{d_v}$ in the row corresponding to v, and has zero entries elsewhere.

Lemma 18. *We note that $\mathcal{L}$ can be written as*

$$\mathcal{L} = SS^*.$$

Lemma 19. *The Matrix $\mathcal{L}$ is hermitian.*

Proof. By spectral theorem. □

Definition 20. Following 19, the eigenvalues of $\mathcal{L}$ are all real and non-negative.

Definition 21. The Dirichlet sum of a graph G for a function $f : \alpha \to \mathbb{R}$ is defined as

$$\sum_{u \sim v} (f(u) - f(v))^2.$$

Definition 22. Let τ denote the constant function which assigns the value 1 on each vertex.

Theorem 23. *$T^{1/2} * \tau$ is an eigenfunction of $\mathcal{L}$ with eigenvalue 0.*

1.3 Basic facts about the spectrum of a graph

Lemma 24. *The sum of the eigenvalues of the graph is at most its number of vertices, that is*

$$\sum_i \lambda_i \leq n.$$

Proof. Follows from considering the trace of $\mathcal{L}$. □

Lemma 25. *In the previous lemma, the equality holds if and only if G has no isolated vertices.*

Lemma 26. *When $n > 1$, we obtain the following upper bound on the second eigenvalue:*

$$\lambda_1 \leq \frac{n}{n-1}.$$

Chapter 2

Connectivity of general graph

Definition 27 (Walk)**.** A walk is a sequence of adjacent vertices. For vertices $u, v \in V$, the walk between them is a sequence starting at u and ending at v.

Definition 28 (Reachable)**.** Two vertices u and v are said to be reachable if there is a walk between them.

Definition 29 (Preconnected)**.** A graph is preconnected if every pair of vertices is reachable from one another.

Definition 30 (Connected)**.** A graph is connected if it is preconnected and contains at least one vertex.

Definition 31 (Edge Boundary)**.** For a subset of vertices $S \subseteq V$, the edge boundary ∂S consists of all edges with exactly one endpoint in S and the other in S^c.

Lemma 32. *If there is a walk going from a vertex $u \in S$ to a vertex $v \in S^c$, then there is at least one edge that goes from S to S^c.*

Lemma 33. *In a connected graph, every set of vertices S different from the empty set and the universal set ($S \neq \emptyset$ and $S \neq V$) has a non-empty edge boundary, that is $\partial S \neq \emptyset$.*

Chapter 3

Isoperimetric problems

Definition 34 (Edge Connection)**.** For two sets of vertices A and B, $E(A, B)$ denotes the set of edges with one endpoint in A and one endpoint in B.

Lemma 35. *For any set of vertices S, the edge boundary of S is equal to the edge boundary of its complement:*

$$\partial S = \partial S^c.$$

Lemma 36. *The edge boundary of a set S is equal to the edge connection between S and S^c:*

$$\partial S = E(S, S^c).$$

Definition 37. For a vertex set S, we define $h_G(S)$ as the ratio of the size of its edge boundary to the minimum volume of S and its complement:

$$h_G(S) = \frac{|E(S, S^c)|}{\min(\mathrm{vol}(S), \mathrm{vol}(S^c))}.$$

Definition 38 (Cheeger Constant)**.** The Cheeger constant of a graph G, denoted h_G, is defined as the minimum of $h_G(S)$ for every set of vertices S with non-zero volume and co-volume:

$$h_G = \inf_{\substack{S \subset V \\ 0 < \min(\mathrm{vol}(S), \mathrm{vol}(S^c))}} h_G(S).$$

Lemma 39. *For any set of vertices S satisfying* $\mathrm{vol}(S) \leq \mathrm{vol}(S^c)$ *and* $0 < \min(\mathrm{vol}(S), \mathrm{vol}(S^c))$*, we have:*

$$h_G \cdot \mathrm{vol}(S) \leq |\partial S|.$$

Lemma 40. *A graph G is connected if and only if its Cheeger constant is strictly positive:*

$$G \text{ is connected} \iff 0 < h_G.$$

Lemma 41. *We derive a simple upper bound for the first non-trivial eigenvalue λ_1 in terms of the Cheeger constant of a connected graph:*

$$\lambda_1 \leq 2h_G.$$